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Björling problem for spacelike, zero mean curvature surfaces in $\mathbb{L}^4$

Antonio C. Asperti[a,*], José Antonio M. Vilhena[b]

[a] *Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, CEP 05311-970, São Paulo, Brazil*
[b] *Universidade Federal do Pará-UFPA, Departamento de Matemática-CCEN, Rua Augusto Corrêa 01, CEP 66075-110, Belém - Pará, Brazil*

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Abstract

In this paper, we extend and solve the Björling-type problem for spacelike, zero mean curvature surfaces in the Lorentz–Minkowski four-dimensional space $\mathbb{L}^4$. As an application we establish symmetry principles for this class of surfaces in $\mathbb{L}^4$ and construct new examples.

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1. Introduction

It is well known that spacelike, zero mean curvature surfaces in $\mathbb{L}^3$ represent locally a maximum for the area integral [15,7] and also that they admit a Weierstrass-

* Corresponding author. Tel.: +55 11 30916176; fax: +55 11 30916183.
E-mail addresses: asperti@ime.usp.br (A.C. Asperti); vilhena@ime.usp.br, javilhena@yahoo.com.br (J.A.M. Vilhena).

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type representation [21,22]. But the spacelike, zero mean curvature surfaces in $\mathbb{L}^4$, represent locally the maximum (resp. minimum) for the area integral, if the normal variation is made in the timelike (resp. spacelike) direction [19]. For these surfaces we also have Weierstrass-type representation [3,12]. An important difference between the global theory of spacelike, zero mean curvature surfaces in $\mathbb{L}^3$ and of the global theory of spacelike, zero mean curvature surfaces in $\mathbb{L}^4$ is established by the so-called *Calabi–Bernstein theorem*. It states that a complete spacelike, zero mean curvature surface in $\mathbb{L}^3$ is a plane [7,8]. However, this result cannot be extended to $\mathbb{L}^n$, $n \geq 4$ [11].

In the three-dimensional Euclidian space $\mathbb{R}^3$, given a real analytic strip (see Section 3), the classical Björling problem [9,16] was proposed by Björling [6] in 1844 and consists of the construction of a minimal surface in $\mathbb{R}^3$ containing the strip in the interior. The solution for this problem was given by Schwarz in [25] by means of a explicit formula in terms of the prescribed strip. This formula gives a beautiful method, besides the Weierstrass representation [24], to construct minimal surfaces with interesting properties. For example, properties of symmetry.

The equivalent problem in the Lorentz–Minkowski three-dimensional space was proposed and solved, using a complex representation formula developed in [1]. The authors introduced the local theory of spacelike, zero mean curvature surfaces in $\mathbb{L}^3$ in a different way of that given in [21,22] through the Weierstrass representation. They constructed new examples of space like, zero mean curvature surfaces, gave alternative proofs of the characterization of the spacelike, zero mean curvature surfaces of revolution and the ruled surfaces in $\mathbb{L}^3$ and proved symmetry principles for those surfaces. We can also find in the work of Gálvez and Mira [13], the version of Björling problem in the hyperbolic three-dimensional space $\mathbb{H}^3$. In that paper the authors constructed the unique mean curvature one surface in $\mathbb{H}^3$ that passes through a given curve with a given unit normal along it, and provide diverse applications.

In Euclidian four-dimensional space, the Björling problem for minimal surfaces was proposed and solved in [4], see also [2], from a complex representation formula. In that work the authors also recovered the symmetry principles of minimal surfaces in $\mathbb{R}^4$ obtained by Eisenhart [10].

In this paper, motivated by results and techniques of [1,4,12], we introduce the local theory of spacelike, zero mean curvature surfaces in $\mathbb{L}^4$, using a complex representation formula – see Theorem 3.1 – that describes the local geometry of these surfaces. This formula is used to solve the Björling problem in $\mathbb{L}^4$, which is illustrated with two examples. As another consequence of Theorem 3.1 we recover the representation formulae of the Björling problem for minimal surfaces in $\mathbb{R}^3$ and spacelike, zero mean curvature surfaces in $\mathbb{L}^3$. We also recover the symmetry principles for these surfaces. Finally, we study the symmetry principles for the spacelike, zero mean curvature surfaces in $\mathbb{L}^4$ and present new examples.

It is not difficult to see that the results in this paper can be extended to spacelike, zero mean curvature surfaces in $\mathbb{L}^n$, $n \geq 4$. Here we restricted the problem to the case $n = 4$ because the formulae and statements are more concise in this case. Also, the case $n = 4$ it is the simplest example of a relativistic spacetime.

2. Preliminaries

Let $\mathbb{L}^4$ denote the four-dimensional Lorentz–Minkowski space, that is, the vector space $\mathbb{R}^4 := \{(x^1, x^2, x^3, x^4) : x^i \in \mathbb{R}\}$ endowed with the Lorentzian metric

$$\langle\, ,\, \rangle := (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 - (\mathrm{d}x^4)^2. \tag{1}$$

Given u, v, w in $\mathbb{L}^4$, we define the vector product $\boxtimes(u, v, w) \in \mathbb{L}^4$ by

$$\langle \boxtimes(u, v, w), x\rangle := -\det(u, v, w, x), \tag{2}$$

which in coordinates takes the form

$$\boxtimes(u, v, w) = \left(\begin{vmatrix} u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \\ u^4 & v^4 & w^4 \end{vmatrix}, - \begin{vmatrix} u^1 & v^1 & w^1 \\ u^3 & v^3 & w^3 \\ u^4 & v^4 & w^4 \end{vmatrix}, \begin{vmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^4 & v^4 & w^4 \end{vmatrix}, \begin{vmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \end{vmatrix} \right).$$

Let $\{e_1, e_2, e_3, e_4\}$ be the canonical basis of $\mathbb{R}^4$. The proof of the following proposition is straightforward.

Proposition 2.1. *The vector product $\boxtimes$ has the following properties*:

(1) $\langle \boxtimes(u, v, w), u\rangle = \langle \boxtimes(u, v, w), v\rangle = \langle \boxtimes(u, v, w), w\rangle = 0$;
(2) $\boxtimes(u, v, e_4) = \hat{u} \times \hat{v}$, *where* $\hat{u} = (u^1, u^2, u^3, 0)$, $\hat{v} = (v^1, v^2, v^3, 0) \in \mathbb{R}^3 \subset \mathbb{L}^4$;
(3) $\boxtimes(u, v, e_2) = \check{u} \times \check{v}$, *where* $\check{u} = (u^1, 0, u^3, u^4)$, $\check{v} = (v^1, 0, v^3, v^4) \in \mathbb{L}^3 \subset \mathbb{L}^4$;
(4) $\boxtimes(u, v, e_1) = -\check{u} \times \check{v}$ *and* $\boxtimes(u, v, e_3) = -\check{u} \times \check{v}$;
(5) $\langle \boxtimes(u_1, u_2, u_3), \boxtimes(v_1, v_2, v_3)\rangle = -\det(\langle u_i, v_j\rangle)$, $1 \leq i; j \leq 3$, $u_i, v_j \in \mathbb{L}^4$; *where* $\times$ *is the cross-product of* $\mathbb{R}^3$ *or of* $\mathbb{L}^3$.

Let $\mathbb{C}^n_1$ be the n-dimensional complex vector space endowed with the *hermitian structure*

$$\ll z, w \gg := \sum_{j=1}^{n-1} z^j \bar{w}^j - z^n \bar{w}^n.$$

We will deal with the following subsets of the *complex projective space* $\mathbb{P}(\mathbb{C}^n_1)$ associated to $\mathbb{P}(\mathbb{C}^n_1)$ (see [14,5,23]):

(1) $\mathbb{CP}^{n-1}_1 := \{z \in \mathbb{C}^n \setminus \{0\} : \ll z, z \gg > 0\}/\mathbb{C}^*$;
(2) $\mathbb{CH}^{n-1} := \{z \in \mathbb{C}^n \setminus \{0\} : \ll z, z \gg < 0\}/\mathbb{C}^*$;
(3) $\partial\mathbb{CH}^{n-1} := \{z \in \mathbb{C}^n \setminus \{0\} : \ll z, z \gg = 0\}/\mathbb{C}^*$.

Denote by $G^+_{2,4}$ the Grassmannian of spacelike 2-planes of $\mathbb{L}^4$ with the induced orientation. Given $u, v \in \mathbb{L}^4$, with $\langle u, u\rangle = \langle v, v\rangle = \lambda^2 > 0$ and $\langle u, v\rangle = 0$, let $\Pi^2 = \mathrm{span}[u, v] \in G^+_{2,4}$. We can identify $G^+_{2,4}$ with $Q^2_1 := \{[z] \in \mathbb{CP}^{n-1}_1 : \ll z, \bar{z} \gg = 0\}$ through the mapping that sends each $\Pi^2 \in G^+_{2,4}$ into $[z] \in Q^2_1$ where $z = u + \mathrm{i}v$. Given $\Pi^2 = \mathrm{span}[u, v] \in G^+_{2,4}$, let $\nu_0 := \boxtimes(u, v, e_4)$ and $\tau_0 := \boxtimes\left(\frac{u}{\lambda}, \frac{v}{\lambda}, \nu_0\right)$; then $\{\nu_0, \tau_0\}$ is a basis for $(\Pi^2)^\perp$.

Proposition 2.2. *Let ν_0 and τ_0 defined as above. We have*:

(1) $\nu_0 = \hat{u} \times \hat{v}$, *where $\times$ is the cross-product in $\mathbb{R}^3 \subset \mathbb{L}^4$*;
(2) $\tau_0 = \lambda^2 e_4 + u^4 u + v^4 v$, *where $\lambda^2 = \langle u, u\rangle$*;
(3) $\langle \nu_0, \nu_0\rangle = \lambda^2(\lambda^2 + (u^4)^2 + (v^4)^2)$ *and* $\langle \tau_0, \tau_0\rangle = -\lambda^2(\lambda^2 + (u^4)^2 + (v^4)^2)$;
(4) *if* $\mu_0 := \sqrt{\lambda^2(\lambda^2 + (u^4)^2 + (v^4)^2)}$, $\tau := \frac{\tau_0}{\mu_0}$ *and* $\nu := \frac{\nu_0}{\mu_0}$, *then* $\left\{\frac{u}{\lambda}, \frac{v}{\lambda}, \nu, \tau\right\}$ *is a positively oriented orthonormal basis of* $\mathbb{L}^4$.

Denote by $G^-_{2,4}$ the Grassmannian of timelike 2-planes of $\mathbb{L}^4$ with the induced orientation. Given $\nu_1, \nu_2 \in \mathbb{L}^4$, with $\langle \nu_1, \nu_1\rangle = -\langle \nu_2, \nu_2\rangle = \lambda^2 > 0$ and $\langle \nu_1, \nu_2\rangle = 0$, let $\Pi^2 = \mathrm{span}[\nu_1, \nu_2] \in G^-_{2,4}$. We can identify, as above, $G^-_{2,4}$ with the real quadric QR, which is defined as the set of classes $[z] \in \partial\mathbb{CH}^{n-1}$ such that $\langle \mathfrak{Re}(z), \mathfrak{Im}(z)\rangle = 0$ and $\mathfrak{Re}(z), \mathfrak{Im}(z)$ are linearly independent.

Definition 2.3. A smooth immersion $X : M^2 \to \mathbb{L}^4$ of a two-dimensional oriented connected manifold is called a *spacelike surface* S in $\mathbb{L}^4$ if the induced metric $\mathrm{d}s^2 := X^*\langle , \rangle$ on M^2 is a Riemannian metric.

Let $(U, z = u + \mathrm{i}v)$ be isothermal coordinates in a neighborhood of a point p in M^2, that is $\langle X_u, X_u\rangle = \langle X_v, X_v\rangle = \lambda^2$ and $\langle X_u, X_v\rangle = 0$. This induces a holomorphic structure on M^2. We define an orthonormal basis $\{\nu, \tau\}$ of $(T_pS)^\perp$ by

$$\nu = \frac{\nu_0}{\mu_0} \quad \text{and} \quad \tau = \frac{\tau_0}{\mu_0}, \tag{3}$$

where

$$\nu_0 = \boxtimes(X_u, X_v, e_4), \qquad \tau_0 = \boxtimes(X_u, X_v, \nu_0), \qquad \mu_0 = \sqrt{\lambda^2(\lambda^2 + (x^4_u)^2 + (x^4_v)^2)}.$$

Observe that ν and τ are, respectively, spacelike and timelike vector fields normal to the surface $S = X(M)$ and it is not hard to see that they are globally defined on S. Also, let $\beta = \{\partial_1, \partial_2, \partial_3, \partial_4\}$ be the local orthonormal frame adapted to S, where

$$\partial_1 = \frac{X_u}{\lambda}, \qquad \partial_2 = \frac{X_v}{\lambda}, \qquad \partial_3 = \nu, \qquad \partial_4 = \tau. \tag{4}$$

As far as we know, the normal frame $\{\nu, \tau\}$ was introduced in [12], where spacelike surfaces in $\mathbb{L}^4$ are extensively studied. Let $\bar{\nabla}$ and ∇ be the Levi–Civita connection of $\mathbb{L}^4$ and $(M^2, \mathrm{d}s^2)$, respectively. The *second fundamental form* of S is defined by $\alpha(V, W) := (\bar{\nabla}_V W)^\perp$ and the *mean curvature vector* by $H_p := \frac{1}{2}\mathrm{tr}(\alpha_p)$ for all $p \in M^2$.

Proposition 2.4. *If $S = X(M)$ is a spacelike surface in $\mathbb{L}^n$, then $\Delta_M X = 2H$.*

Proof. See [12]. □

Remark 2.5. A spacelike surface S in $\mathbb{L}^4$ has zero mean curvature if $H = 0$ for all points of S.

Let $S = X(M)$ a spacelike surface in $\mathbb{L}^4$ defined in terms of local isothermal coordinates $(U, z = u + \mathrm{i}v)$ of M^2, and define the complex functions

$$\varphi^k := \frac{\partial x^k}{\partial u} - \mathrm{i}\frac{\partial x^k}{\partial v}, \quad k = 1, 2, 3, 4. \tag{5}$$

It is not hard see that

$$(\varphi^1)^2 + (\varphi^2)^2 + (\varphi^3)^2 - (\varphi^4)^2 = 0, \qquad |\varphi^1|^2 + |\varphi^2|^2 + |\varphi^3|^2 - |\varphi^4|^2 = 2\lambda^2 > 0.$$

The induced metric on M is $\mathrm{d}s^2 = \lambda^2|\mathrm{d}z|^2$ and the complex 1-forms $\omega^k := \varphi^k\,\mathrm{d}z$ are globally defined on M. Now if S is a spacelike, zero mean curvature surface, it follows from Proposition 2.4 that ω^k is holomorphic. Thus, S can be represented as

$$X(z) = \Re\mathfrak{e}\int_{z_0}^{z} \omega + k_0, \quad \text{where } \omega = (\omega^1, \omega^2, \omega^3, \omega^4) \text{ and } z_0, z \in M. \tag{6}$$

The converse also holds.

Theorem 2.6. *Let M^2 be a connected Riemann surface and $\omega = (\omega^1, \omega^2, \omega^3, \omega^4)$ a holomorphic* 1*-form with values in $\mathbb{C}^4$ globally defined on M^2 satisfying*

(1) $\ll \omega, \overline{\omega} \gg \equiv 0$,
(2) $\ll \omega, \omega \gg > 0, \forall p \in M^2$,
(3) $\Re\mathfrak{e}\int_\gamma \omega = 0$, *for all closed path γ on M^2.*

Then the map $X : M^2 \to \mathbb{L}^4$ given by the Eq. (6) *defines a spacelike, zero mean curvature surface in $\mathbb{L}^4$.*

For the proof see [12]. The *Gauss map* $G : M^2 \to Q_1^2$ of a spacelike surface $S = X(M)$ in $\mathbb{L}^4$ is defined locally by $G(z) = [\overline{\Phi(z)}]$, with $X_z = \psi\Phi$ for some function $\psi : M^2 \to \mathbb{C}$ and $\Phi = (\phi^1, \phi^2, \phi^3, \phi^4)$, for more details see [18,12]. Let $a(z), b(z)$ be the complex valued functions defined on M^2 by

$$a(z) := \frac{-\phi^3 + \phi^4}{\phi^1 - \mathrm{i}\phi^2}, \qquad b(z) := \frac{\phi^3 + \phi^4}{\phi^1 - \mathrm{i}\phi^2}. \tag{7}$$

We have that

$$\Phi(z) = \mu(1 + ab, \mathrm{i}(1 - ab), a - b, a + b). \tag{8}$$

It follows from (3) and (8) that

$$\tau(z) = \frac{1}{|1-a\bar{b}|\sqrt{(1+|a|^2)(1+|b|^2)}} \begin{bmatrix} (1+|b|^2)\mathfrak{Re}(a) + (1+|a|^2)\mathfrak{Re}(b) \\ (1+|b|^2)\mathfrak{Im}(a) + (1+|a|^2)\mathfrak{Im}(b) \\ |a|^2 - |b|^2 \\ (1+|a|^2)(1+|b|^2) \end{bmatrix},$$

$$\nu(z) = \frac{1}{|1-a\bar{b}|\sqrt{(1+|a|^2)(1+|b|^2)}} \begin{bmatrix} (1+|b|^2)\mathfrak{Re}(a) - (1+|a|^2)\mathfrak{Re}(b) \\ (1+|b|^2)\mathfrak{Im}(a) - (1+|a|^2)\mathfrak{Im}(b) \\ |a|^2|b|^2 - 1 \\ 0 \end{bmatrix}.$$

For more details see [12]. Let $A : M^2 \to \mathbb{C}^4$ be the complex map defined by

$$A(z) := \nu(z) + \mathrm{i}\tau(z), \tag{9}$$

and observe that $[A(z)] \in QR$.

3. Main results

Now we are able to propose and solve the Björling problem for spacelike, zero mean curvature surfaces in $\mathbb{L}^4$. Let $c : I \subseteq \mathbb{R} \to \mathbb{L}^4$ be a regular real analytic spacelike curve in $\mathbb{L}^4$ and let $n : I \to \mathbb{C}^4$ be a real analytic vector field along c (that is, $\mathfrak{Re}(n), \mathfrak{Im}(n) : I \to \mathbb{L}^4$ are vector fields along c) such that $\langle c'(s), \mathfrak{Re}(n)\rangle = 0 = \langle c'(s), \mathfrak{Im}(n)\rangle$, $\langle \mathfrak{Re}(n), \mathfrak{Re}(n)\rangle = -\langle \mathfrak{Im}(n), \mathfrak{Im}(n)\rangle = 1$ and $\mathfrak{Im}(n)$ is future directed for all $s \in I$. In analogy with [1,9], we call such a pair (c, n) a *analytical strip* in $\mathbb{L}^4$. The problem is then to find a spacelike, zero mean curvature surface S defined by $X : \Omega \subseteq \mathbb{C} \to \mathbb{L}^4$ with $I \subset \Omega$, such that

(1) $X(u, 0) = c(u)$,
(2) $A(u, 0) = n(u), \forall\, u \in I$.

It is easy to see that if $X : \Omega \subseteq \mathbb{C} \to \mathbb{L}^4$ is a spacelike, zero mean curvature surface in $\mathbb{L}^4$, then $c(u) := X(u, 0)$ and $n(u) := A(u, 0)$ satisfy the above data and, in particular, they are real analytic. Then there exist holomorphic extensions $c(z)$ and $n(z)$ and these extensions are unique by the *identity theorem* for analytic functions (see [22, p. 87]). In this situation, we can explicitly recover $X(z)$ from c and n by means of a unique complex representation formula.

Theorem 3.1. *Let S be a spacelike, zero mean curvature surface in $\mathbb{L}^4$ given by $X : U \subseteq \mathbb{C} \to \mathbb{L}^4$. Define the curve $c(u) := X(u, 0)$ and the vector field $n(u) := A(u, 0)$ along c, on a real interval $I \subset U$. Choose any simply connected open set $\Omega \subseteq U$ containing I, over which we can define holomorphic extensions $c(z)$ and $n(z)$ of c and n. Then, for all $z \in \Omega$*

it holds

$$X(z) = \mathfrak{Re}\left(c(z) + \mathrm{i}\int_{s_0}^{z} \boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w))\,\mathrm{d}w\right), \tag{10}$$

where s_0 is a arbitrary fixed point of I and the integral is taken along an arbitrary path in Ω joining s_0 and z.

Proof. Since S has zero mean curvature, the complex function $\Psi : U \to \mathbb{C}^4$ defined by (5)

$$\Psi = 2\frac{\partial X}{\partial z} \text{ with } \Psi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4),$$

is holomorphic in U and by (6) we can write

$$X(z) = \mathfrak{Re}\int_{z_0}^{z} \Psi\,\mathrm{d}z + k_0, \tag{11}$$

where $k_0 \in \mathbb{L}^4$ is a suitable constant such that $X(u, 0) = c(u)$ for all $u \in I$. Let $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ be the local orthonormal frame adapted to S given in (4). Now write $\boxtimes$ in this basis,

$$\boxtimes(\partial_3, \partial_4, \partial_1) = \langle\boxtimes(\partial_3, \partial_4, \partial_1), \partial_2\rangle\partial_2 = -\det(\partial_1, \partial_2, \partial_3, \partial_4)\partial_2 = -\partial_2,$$

and since $X_v = \lambda\partial_2$, we have

$$\Psi(z) = X_u(z) - \mathrm{i}X_v(z) = X_u + \mathrm{i}\boxtimes(\nu(z), \tau(z), X_u(z)) \tag{12}$$

in isothermal coordinates (U, $z = u + \mathrm{i}v$). Restricting $\Psi(z)$ to I and using the definition of c, n we obtain

$$\begin{aligned}\Psi(u, 0) &= X_u(u, 0) + \mathrm{i}\boxtimes(\nu(u, 0), \tau(u, 0), X_u(u, 0))\\ &= c'(u) + \mathrm{i}\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u)).\end{aligned}$$

Since these functions are real analytic, we can extend them to two holomorphic functions $\Psi(z)$, $c'(z) + i\boxtimes(\mathfrak{Re}(n(z)), \mathfrak{Im}(n(z)), c'(z))$ on a simply connected open set $\Omega \subseteq U$ and they coincide on $I \subset \Omega$. Hence by the *identity theorem* for analytic functions it follows that

$$\Psi(z) = c'(z) + \mathrm{i}\boxtimes(\mathfrak{Re}(n(z)), \mathfrak{Im}(n(z)), c'(z)), \quad \forall z \in \Omega.$$

Therefore

$$\Gamma(z) := c(z) + \mathrm{i}\int_{s_0}^{z} \boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w))\,\mathrm{d}w, \quad \forall z \in \Omega$$

is well defined on Ω and obviously is the primitive of the holomorphic mapping $\Psi(z)$. Thus, (11) yields

$$X(z) = \mathfrak{Re}\left(c(z) + \mathrm{i}\int_{s_0}^{z} \boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w))\,\mathrm{d}w\right).$$

This completes the proof of the theorem. □

Remark 3.2. We can choose any $s_0 \in I$ in (10) and the values of $X(z)$ will remain the same, since $c'(z)$, $\mathfrak{Re}(n(z))$, $\mathfrak{Im}(n(z))$ all take real values in $I \in \Omega$.

Using the complex representation formula given in (10), we now show that the Björling problem has a unique solution.

Theorem 3.3. *There exists a unique solution $X : \Omega \to \mathbb{L}^4$ to the Björling problem for spacelike, zero mean curvature surfaces in $\mathbb{L}^4$, which is given by*

$$X(z) = \mathfrak{Re}\left(c(z) + \mathrm{i}\int_{s_0}^{z} \boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w))\,\mathrm{d}w\right), \tag{13}$$

with $w = u + \mathrm{i}v \in \Omega$, $s_0 \in I$, where Ω is a simply connected open subset of $\mathbb{C}$ containing the real interval I and for which c, n admit holomorphic extensions $c(z)$, $n(z)$.

Proof. Define the holomorphic curve $\Psi : \Omega \subseteq \mathbb{C} \to \mathbb{C}^4$ by

$$\Psi(z) = c'(z) + \mathrm{i}\boxtimes(\mathfrak{Re}(n(z)), \mathfrak{Im}(n(z)), c'(z)), \quad \forall z \in \Omega. \tag{14}$$

where Ω is a simply connected open subset of $\mathbb{C}$ containing I on which the holomorphic extensions $c(z)$, $n(z)$ exist. Since by Proposition 2.1, $c'(u)$ and $\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ are orthogonal and have the same length, it follows that

$$(\varphi^1(u,0))^2 + (\varphi^2(u,0))^2 + (\varphi^3(u,0))^2 - (\varphi^4(u,0))^2 = 0, \forall u \in I.$$

We also have that

$$|\varphi^1(u,0)|^2 + |\varphi^2(u,0)|^2 + |\varphi^3(u,0)|^2 - |\varphi^4(u,0)|^2 = 2\langle c'(u), c'(u)\rangle > 0.$$

Thus

$$(\varphi^1(z))^2 + (\varphi^2(z))^2 + (\varphi^3(z))^2 - (\varphi^4(z))^2 = 0,$$
$$|\varphi^1(z)|^2 + |\varphi^2(z)|^2 + |\varphi^3(z)|^2 - |\varphi^4(z)|^2 > 0,$$

for all $z \in \Omega$. Moreover, the holomorphic curve Ψ has no real periods for Ω is simply connected. Therefore by Theorem 2.6, $X(z) = \mathfrak{Re}\int_{s_0}^{z} \Psi(w)\,\mathrm{d}w$ defines a spacelike, zero mean curvature surface $S = X(\Omega)$ in $\mathbb{L}^4$, where Ψ is given by (14) and $s_0 \in I$. Now we shall check that this surface satisfies the Björling conditions $X(u,0) = c(u)$ and $A(u,0) = n(u)$. The verification of the first condition is easy, since $\boxtimes(\mathfrak{Re}(n), \mathfrak{I}m(n), c')$ is real when restricted to I. To check the second condition, first recall that $\Psi = 2(\partial X/\partial z)$. So it follows from (14) that, restricted to I, we have

$$X_u(u,0) = c'(u) \quad \text{and} \quad X_v(u,0) = -\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u)).$$

On the other hand, from (12) we have

$$X_v(u,0) = -\boxtimes(\nu(u,0), \tau(u,0), c'(u)).$$

Since $\mathfrak{Im}(n(u))$ is future directed it follows that $\mathfrak{Re}(n(u)) = \nu(u,0)$ and $\mathfrak{Im}(n(u)) = \tau(u,0)$. At last we will prove the uniqueness, which is to be understood in the following sense: if $\tilde{X}(u,v)$, $z = u + \mathrm{i}v \in \tilde{\Omega}$ is another solution, then $X(u,v) = \tilde{X}(u,v)$ for $z = u + \mathrm{i}v \in \Omega \cap \tilde{\Omega}$. In fact, any pair of solutions X, $\tilde{X}$ to the Björling problem coincide on a real interval $I \subset \Omega \cap \widetilde{\Omega}$, and since both are analytic they must coincide on $\Omega \cap \tilde{\Omega}$. This completes the proof of the theorem. □

Remark 3.4. Observe that the unicity in the above theorem is only referred to spacelike, zero mean curvature surfaces $X : \Omega \subseteq \mathbb{C} \to \mathbb{L}^4$ satisfying $X(u,0) = c(u)$ and $A(u,0) = n(u)$. Actually a little more can be proven: given an analytic strip (c, n) in $\mathbb{L}^4$, there exists a unique spacelike, zero mean curvature immersion $X : M^2 \to \mathbb{L}^4$ whose image contains $c(I)$ and A restricted to c is n. The existence part of this statement follows from Theorem 3.3. For the unicity part, we refer to Corollary 3.4 of [1]. There unicity is proven for analytic strips in $\mathbb{L}^3$ and spacelike, zero mean curvature surfaces in $\mathbb{L}^3$, but their arguments work in our case as well.

Example 3.5. Consider

$$c(s) = (s - s^3, 0, s^2, 0) \in \mathbb{L}^4,$$
$$n(s) = \frac{1}{(1 - 2s^2 + 9s^4)^{1/2}}(2s, -2\sqrt{2}si, -(1 - 3s^2), (1 + 3s^2)i) \in \mathbb{C}^4,$$

for all $s \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$\boxtimes(\mathfrak{Re}(n(s)), \mathfrak{Im}(n(s)), c'(s)) = (0, 1 + 3s^2, 0, -2\sqrt{2}s),$$

whose holomorphic extension is

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (0, 1 + 3w^2, 0, -2\sqrt{2}w).$$

Thus

$$X(z) := \mathfrak{Re}((z - z^3, 0, z^2, 0)) - \mathfrak{Im}((0, z + z^3 - (s_0 + s_0^3), 0, -\sqrt{2}z^2 + \sqrt{2}s_0^2))$$

and therefore, the solution of the Björling problem for the given strip is

$$X(z) = (u + 3uv^2 - u^3, -v - 3u^2v + v^3, u^2 - v^2, 2\sqrt{2}uv),$$

with $z = u + \mathrm{i}v \in \mathbb{C}$.

Example 3.6. Consider

$$c(s) = (1 + \cos(s), 0, \sin(s), 2\sin(s/2)) \in \mathbb{L}^4,$$
$$n(s) = (\cos(s), 0, \sin(s), 0) + \mathrm{i}(-1 - \cos(s), 0, \cos(s)\cot(s/2), \csc(s/2)) \in \mathbb{C}^4,$$

for all $s \in (0, 2\pi)$. By similar calculations,

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (0, \sin(z/2), 0, 0).$$

Then

$$X(z) := \mathfrak{Re}((1 + \cos(z), 0, \sin(z), 2\sin(z/2))) - \mathfrak{Im}(-2\cos(z/2) + 2\cos(s_0/2))$$

and therefore, the solution of the Björling problem for the given strip is

$$\begin{aligned} X(z) = (&1 + \cos(u)\cosh(v), \\ &-2\sin(u/2)\sinh(v/2), \cosh(v)\sin(u), 2\cosh(v/2)\sin(u/2)), \end{aligned}$$

with $z = u + \mathrm{i}v$, where $u \in (0, 2\pi)$ and $v \in \mathbb{R}$.

As consequences of Theorem 3.3 we recover the classical Björling problem for $\mathbb{R}^3$ and also the Björling problem for $\mathbb{L}^3$, see [1].

Corollary 3.7. *Let $c : I \to \mathbb{R}^3$, $\mathbb{R}^3 \equiv \{x^4 = 0\} \subset \mathbb{L}^4$, be a regular real analytic curve and let $n : I \to \mathbb{C}^4$ be a real analytic vector field along c such that $n(s) = \xi(s) + \mathrm{i}e_4$, where $\xi(s) \in \mathbb{R}^3$ is a unitary vector field satisfying $\langle c'(s), \xi(s)\rangle = 0$ for all $s \in I$. Then there exists a unique solution to the Björling problem for minimal surfaces in $\mathbb{R}^3$, which is given by*

$$X(z) = \mathfrak{Re}\left\{c(z) - \mathrm{i}\int_{s_0}^{z} (\xi(w) \times c'(w))\,\mathrm{d}w\right\}, \tag{15}$$

where $w = u + \mathrm{i}v \in \Omega$, $s_0 \in I$, Ω is a simply connected open set of $\mathbb{C}$ containing I and $\times$ is the cross-product of $\mathbb{R}^3$.

Proof. From Theorem 3.3 it follows that the solution to the Björling problem is given by

$$\begin{aligned} X(z) &= \mathfrak{Re}\left(c(z) + \mathrm{i}\int_{s_0}^{z} \boxtimes(\xi(w), e_4, c'(w))\,\mathrm{d}w\right) \\ &= \mathfrak{Re}\left(c(z) - \mathrm{i}\int_{s_0}^{z} \boxtimes(\xi(w), c'(w), e_4)\,\mathrm{d}w\right). \end{aligned}$$

Hence, from Proposition 2.1 item 2 we have

$$X(z) = \mathfrak{Re}\left(c(z) - \mathrm{i}\int_{s_0}^{z} \hat{\xi}(w) \times \hat{c}'(w)\,\mathrm{d}w\right) = \mathfrak{Re}\left(c(z) - \mathrm{i}\int_{s_0}^{z} \xi(w) \times c'(w)\,\mathrm{d}w\right).$$

Corollary 3.8. *Let $c : I \to \mathbb{L}^3, \mathbb{L}^3 \equiv \{x^2 = 0\} \subset \mathbb{L}^4$, be a regular real analytic spacelike curve and let $n : I \to \mathbb{C}^4$ be a real analytic vector field along c of the form $n(s) = e_2 + \mathrm{i}V(s)$, where $V(s) \in \mathbb{L}^3$ is a future directed, timelike unitary vector field such that $\langle c'(s), V(s)\rangle = 0$ for all $s \in I$. Then there exists a unique solution to the Björling problem for spacelike, zero mean curvature surfaces in $\mathbb{L}^3$, which is given by*

$$X(z) = \mathfrak{Re}\left\{c(z) + \mathrm{i}\int_{s_0}^{z} (V(w) \times c'(w))\,\mathrm{d}w\right\}, \tag{16}$$

where $w = u + \mathrm{i}v \in \Omega$, $s_0 \in I$, Ω is a simply connected open set of $\mathbb{C}$ containing I and $\times$ is the cross-product in $\mathbb{L}^3$.

Proof. From Theorem 3.3 if follows that the solution to the Björling problem is given by

$$\begin{aligned} X(z) &= \mathfrak{Re}\left(c(z) + \mathrm{i}\int_{s_0}^{z} \boxtimes(e_2, V(w), c'(w))\,\mathrm{d}w\right) \\ &= \mathfrak{Re}\left(c(z) + \mathrm{i}\int_{s_0}^{z} \boxtimes(V(w), c'(w), e_2)\,\mathrm{d}w\right). \end{aligned}$$

Hence, from Proposition 2.1 item 3 we have

$$X(z) = \mathfrak{Re}\left(c(z) + i\int_{s_0}^{z} \check{V}(w) \times \check{c}'(w)\,\mathrm{d}w\right) = \mathfrak{Re}\left(c(z) + i\int_{s_0}^{z} V(w) \times c'(w)\,\mathrm{d}w\right).$$

□

4. Symmetries

Now, we will study the symmetries of the spacelike, zero mean curvature surfaces in $\mathbb{L}^4$ via the complex representation formula of the Björling problem for spacelike, zero mean curvature surfaces. In order to do so we fix the following notation. Let $f(z) = x(z) + \mathrm{i}y(z)$, where $x(z)$, $y(z)$ are real-valued functions defined on the open set Ω of $\mathbb{C}$. If $x(z)$ is harmonic and $f(z)$ is holomorphic in Ω, then $x(\overline{z})$ is harmonic and $\overline{f(\overline{z})}$ is holomorphic as a function of z in the open set $\Omega^* := \{\overline{z} : z \in \Omega\}$. Note that, Ω is symmetric if only if $\Omega = \Omega^*$. We also have that, if $I \subset \Omega$, f is holomorphic in Ω and f restrict to I take only real values, then $f(z) = \overline{f(\overline{z})}$ on $I \subset \Omega \cap \Omega^*$. Therefore, $f(z)$ can be holomorphically extended to $\Omega \cup \Omega^*$.

Proposition 4.1. *Let $X : \Omega \subseteq \mathbb{C} \to \mathbb{L}^4$ be the solution of the Björling problem, for a given strip (c, n)* in $\mathbb{L}^4$, *where Ω is a symmetric simply connected open set containing the* *real interval I and for which c and n admit holomorphic extensions* $c(z)$ and $n(z)$, where $z = u + \mathrm{i}v \in \Omega$. Then for all $z \in \Omega$ we have

$$X(\overline{z}) = \mathfrak{Re}\left\{c(z) - \mathrm{i}\int_{s_0}^{z} \boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w))\,\mathrm{d}w\right\}. \tag{17}$$

Proof. The surface $\tilde{S} = \tilde{X}(\Omega)$ given by $\tilde{X}(u, v) := X(u, -v)$, clearly satisfies $\tilde{X}_{uu}(u, v) = X_{uu}(u, -v)$, $\tilde{X}_{vv}(u, v) = X_{vv}(u, -v)$ and still is a spacelike, zero mean curvature surface in $\mathbb{L}^4$. Associated to $\tilde{X}$, let $\tilde{A}(u, v) := \tilde{\nu}(u, v) + \mathrm{i}\tilde{\tau}(u, v)$. From Proposition 2.2 and the definition of $\boxtimes$, we have that

$$\tilde{\tau}_0(u, v) = (\lambda^2 e_4 + x_u^4 X_u + x_v^4 X_v)(u, -v),$$
$$\tilde{\nu}_0(u, v) = - \boxtimes (e_4, X_u(u, -v), X_v(u, -v)),$$

and hence $\tilde{\tau}(u, v) = \tau(u, -v)$ and $\tilde{\nu}(u, v) = -\nu(u, -v)$. Therefore,

$$\widetilde{A}(u, v) = -\overline{A(u, -v)}. \tag{18}$$

This implies that $\widetilde{A}(u, 0) = -\overline{A(u, 0)} = -\overline{n(u)}$ and $\tilde{X}(u, 0) = X(u, 0) = c(u)$. Hence $\tilde{X}$ is a solution of the Björling problem for $\tilde{c} = c$, $\tilde{n} = -\overline{n}$ and then $\tilde{X}(z) = \mathfrak{Re}\int_{s_0}^{z} \tilde{\Psi}(w)\,\mathrm{d}w$, where $\tilde{\Psi}(z) = \tilde{X}_u + \mathrm{i} \boxtimes (\tilde{\nu}(z), \tilde{\tau}(z), \tilde{X}_u(z))$, see (13). Restricting $\tilde{\Psi}(z)$ to I and using (18) we obtain

$$\begin{aligned}\tilde{\Psi}(u, 0) &= X_u(u, 0) + \mathrm{i} \boxtimes (-\nu(u, 0), \tau(u, 0), X_u(u, 0)) \\ &= c'(u) - \mathrm{i} \boxtimes (\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u)).\end{aligned}$$

When we extend these functions to Ω^*, the result follows. □

The proofs of the following corollaries are analogous to those of Corollaries 3.7 and 3.8.

Corollary 4.2. *Under the hypothesis of* Proposition 4.1, *if* $S = X(\Omega) \subset \mathbb{R}^3 \equiv \{x^4 = 0\}$ *and n is of the form* $n(s) = \xi(s) + \mathrm{i}e_4$, *with* $\xi(s) \in \mathbb{R}^3$ *unitary such that* $\langle c'(s), \xi(s)\rangle = 0$ *for all* $s \in I$, *then*

$$X(\overline{z}) = \mathfrak{Re}\left\{c(z) + \mathrm{i}\int_{s_0}^{z} (\xi(w) \times c'(w))\,\mathrm{d}w\right\}, \quad \textit{for all } z \in \Omega. \tag{19}$$

Corollary 4.3. *Under the hypothesis of* Proposition 4.1, *if* $S = X(\Omega) \subset \mathbb{L}^3 \equiv \{x^2 = 0\}$ *and n is of the form* $n(s) = e_2 + \mathrm{i}V(s)$, *with* $V(s) \in \mathbb{L}^3$ *unitary, future directed, timelike and such that* $\langle c'(s), V(s)\rangle = 0$ *for all* $s \in I$, *then*

$$X(\overline{z}) = \mathfrak{Re}\left\{c(z) - \mathrm{i}\int_{s_0}^{z} (V(w) \times c'(w))\,\mathrm{d}w\right\}, \quad \textit{for all } z \in \Omega. \tag{20}$$

Remark 4.4. Using the formulae (15) and (19), it is not difficult to recover the two symmetry principles discovered by Schwarz for minimal surfaces in $\mathbb{R}^3$(see [11, p. 123]). Also, by using (16) and (20), we can recover the two symmetry principles for spacelike, zero mean curvature surfaces in $\mathbb{L}^3$ given in [2, Theorem 3.10].

Now using (13) and (17) we will derive three symmetry principles for spacelike, zero mean curvature surfaces in $\mathbb{L}^4$. They were motivated by the works of Schwarz and [1] above mentioned. Before going to it, we have the following definitions.

Definition 4.5. Let Π^k be a k-plane in $\mathbb{L}^4$. Assume that Π^k is spacelike if $k = 1$; Π^k is spacelike, timelike or degenerate if $k = 2$; Π^k is timelike if $k = 3$. Under those conditions, we say that Π^k is a k-plane of symmetry of a spacelike surface $X : M^2 \to \mathbb{L}^4$ if for all $p \in M^2$ there exists a certain $q \in M^2$ such that $X(p)$, $X(q)$ are symmetric with respect to Π^k, that is, such that $(X(q) + X(p))/2 \in \Pi^k$ and $X(q) - X(p)$ is perpendicular to Π^k.

Theorem 4.6. *Let S be a spacelike, zero mean curvature surface in L^4, given by $X : U \subseteq \mathbb{C} \to \mathbb{L}^4$. Then we have*:

(1) *every spacelike straight line contained in S is an axis of symmetry of S*;
(2) *if S intersects any timelike or spacelike* 2*-plane Π^2, orthogonally along a curve regular of S, then Π^2 is a plane of symmetry of S*;
(3) *if S intersects any timelike* 3*-space Π^3, orthogonally along a curve regular of S, then Π^3 is a* 3*-plane of symmetry of S.*

Before going through the proof, it is convenient to make the following observation. Suppose for instance that the spacelike, zero mean curvature surface S contains a segment of line L, which, we may assume is a portion of the x^1-axis. Then it is possible to define isothermal coordinates $z = u + iv$ in a neighborhood of L so that $X(u, 0)$ parametrizes L, see [17]. Analogous observations are in place in case S intersects orthogonally the x^1, x^4-plane, or the x^1, x^2-plane or the 3-space $\{x^3 = 0\}$. With this in mind, it is not difficult to see that Theorem 4.6 is now a consequence of the following lemma.

Lemma 4.7. *Let S be a spacelike, zero mean curvature surface in $\mathbb{L}^4$, given by $X : \Omega \subseteq \mathbb{C} \to \mathbb{L}^4$, with Ω is symmetric and simply connected.*

(1) *If, for all $u \in I$, the curve $c(u) = X(u, 0)$, is contained in the x^1-axis, then*
$$X(u, -v) = (x^1(u, v), -x^2(u, v), -x^3(u, v), -x^4(u, v)). \tag{21}$$
(2) *If, for all $u \in I$, the curve $c(u) = X(u, 0)$, is contained in the timelike x^1, x^4-plane Π^2, and if the surface S intersects Π^2 orthogonally along c, then*
$$X(u, -v) = (x^1(u, v), -x^2(u, v), -x^3(u, v), x^4(u, v)). \tag{22}$$
(3) *If, for all $u \in I$, the curve $c(u) = X(u, 0)$, is contained in the spacelike x^1, x^2-plane Π^2, and if the surface S intersects Π^2 orthogonally along c, then*
$$X(u, -v) = (x^1(u, v), x^2(u, v), -x^3(u, v), -x^4(u, v)). \tag{23}$$
(4) *If, for all $u \in I$, the curve $c(u) = X(u, 0)$, is contained in the timelike* 3*-space $\Pi^3 = \{x^2 = 0\}$, and if the surface S intersects Π^3 orthogonally along c, then*
$$X(u, -v) = (x^1(u, v), -x^2(u, v), x^3(u, v), x^4(u, v)). \tag{24}$$

Proof.

(1) Set $c(u) := X(u, 0)$ and $n(u) := A(u, 0)$. By hypothesis, it follows that $c(u) = (c^1(u), 0, 0, 0)$, $\mathfrak{Re}(n(u)) = (0, \nu^2(u, 0), \nu^3(u, 0), \nu^4(u, 0))$ and $\mathfrak{Im}(n(u)) =$

$(0, \tau^2(u,0), \tau^3(u,0), \tau^4(u,0))$. Hence, by a straightforward calculation we have that $\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ is of the form $(0, \boxtimes^2(u), \boxtimes^3(u), \boxtimes^4(u))$. On account of (13), (17) it follows, respectively, that

$$X(z) = \left(\mathfrak{Re}(c^1(z)), -\mathfrak{Im}\int_{s_0}^{z}\boxtimes^2(w)\,\mathrm{d}w,\right.$$
$$\left. -\mathfrak{Im}\int_{s_0}^{z}\boxtimes^3(w)\,\mathrm{d}w, -\mathfrak{Im}\int_{s_0}^{z}\boxtimes^4(w)\,\mathrm{d}w\right),$$
$$X(\bar{z}) = \left(\mathfrak{Re}(c^1(z)), \mathfrak{Im}\int_{s_0}^{z}\boxtimes^2(w)\,\mathrm{d}w, \mathfrak{Im}\int_{s_0}^{z}\boxtimes^3(w)\,\mathrm{d}w, \mathfrak{Im}\int_{s_0}^{z}\boxtimes^4(w)\,\mathrm{d}w\right),$$

which proves (21).

(2) Since by hypothesis, S intersects $\Pi^2 = \{x^2 = 0, x^3 = 0\}$ orthogonally at $c(u) := X(u,0)$, it follows that $c(u) = (c^1(u), 0, 0, c^4(u))$. Now recall that the 2-plane P^2 generated by $\mathfrak{Re}(n(u))$ and $\mathfrak{Im}(n(u))$ is orthogonal to $T_{c(u)}S$ along c. It follows that $\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ is of the form $(0, \boxtimes^2(u), \boxtimes^3(u), 0)$. On account of (13) and (17) we then arrive at the formula (22).

(3) The proof is analogous to item (2).

(4) The hypothesis implies that $c(u) = (c^1(u), 0, c^3(u), c^4(u))$. Since S intersects Π^3 orthogonally, we have that $X_v(u,0) \in (\Pi^3)^\perp$ and therefore $X_v(u,0)$ is parallel to the unitary vector e_2 which is normal to Π^3. Then $\mathfrak{Re}(n(u))$ and $\mathfrak{Im}(n(u))$ lie in Π^3, which implies that the second component of both vectors are equal to zero. Hence $\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ is of the form $(0, \boxtimes^2, 0, 0)$. Therefore, in conjunction with (13) and (17) we obtain (24). □

If in Theorem 4.1 Π^2 is a degenerate two plane, we have the following proposition.

Proposition 4.8. *Let $X : \Omega \subseteq \mathbb{C} \to \mathbb{L}^4$ be a spacelike, zero mean curvature surface, with Ω symmetric, simply connected and assume that $S = X(\Omega)$ intersects the degenerate 2-plane $\Pi^2 = [e_1 + e_4, e_2]$ orthogonally along the curve $c(u) = X(u,0)$. Then S is contained in the degenerate 3-space $\Pi^3 = [e_1 + e_4, e_2, e_3]$. Moreover Π^2 is a plane of symmetry for S if and only if $X_v(u,0)$ is a multiple of e_3.*

Proof. Consider the basis $\mathcal{F} = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ of $\mathbb{L}^4$, where $\epsilon_1 = \frac{\sqrt{2}}{2}(e_1 + e_4)$, $\epsilon_2 = \frac{\sqrt{2}}{2}(e_1 - e_4)$, $\epsilon_3 = e_2$, $\epsilon_4 = e_3$ and observe that $\Pi^2 = [\epsilon_1, \epsilon_3]$. It is clear that $c(u) = X(u,0)$ is of the form $c(s) = (c^1(s), c^2(s), 0, c^1(s))$. Since the 2-plane $P^2 = [\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u))]$ is orthogonal to $T_{c(u)}S$ along c, is follows that $-X_v(u,0) = \boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ is of the form $(\boxtimes^1(u), 0, \boxtimes^3(u), \boxtimes^1(u))$. By the same arguments as before, we obtain that

$$X(z) = \left(\mathfrak{Re}(c^1(z)) - \mathfrak{Im}\int_{s_0}^{z}\boxtimes^1(w)\,\mathrm{d}w, \mathfrak{Re}(c^2(z)),\right.$$
$$\left. -\mathfrak{Im}\int_{s_0}^{z}\boxtimes^3(w)\,\mathrm{d}w, \mathfrak{Re}(c^1(z)) - \mathfrak{Im}\int_{s_0}^{z}\boxtimes^1(w)\,\mathrm{d}w\right),$$

$$X(\overline{z}) = \left(\mathfrak{Re}(c^1(z)) + \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)\,\mathrm{d}w, \mathfrak{Re}(c^2(z)), \mathfrak{Im}\int_{s_0}^{z} \boxtimes^3(w)\,\mathrm{d}w, \mathfrak{Re}(c^1(z))\right.$$
$$\left. + \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)\,\mathrm{d}w\right),$$

which written in the basis $\mathcal{F}$ gives, respectively

$$X(z) = \left(\mathfrak{Re}(c^1(z)) - \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)\,\mathrm{d}w, 0, \mathfrak{Re}(c^3(z)), -\mathfrak{Im}\int_{s_0}^{z} \boxtimes^4(w)\,\mathrm{d}w, \right)_{\mathcal{F}},$$
$$X(\overline{z}) = \left(\mathfrak{Re}(c^1(z)) + \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)\,\mathrm{d}w, 0, \mathfrak{Re}(c^3(z)), \mathfrak{Im}\int_{s_0}^{z} \boxtimes^4(w)\,\mathrm{d}w, \right)_{\mathcal{F}}.$$

The first part is clear and S is symmetric with respect to Π^2 if and only if $\mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)\,\mathrm{d}w = 0$, that is, $\int_{s_0}^{z} \boxtimes^1(w)\,\mathrm{d}w \equiv 0$ and the last claim follows.

Remark 4.9.

(1) It is not difficult to see that Lemma 4.7 and Proposition 4.8 hold without the simply connectivity assumption.
(2) Observe that if Π^3 is spacelike or degenerate, then there is no spacelike vector orthogonal to Π^3 in $\mathbb{L}^4$. Therefore the symmetry problem of spacelike, zero mean curvature surfaces in not defined is these cases.

Example 4.10. Consider

$$c(s) = (0, s, 0, 0) \in \mathbb{L}^4,$$
$$n(s) = \left(\frac{e^{-s}}{\sqrt{4+e^{-2s}}}, 0, -\frac{2}{\sqrt{4+e^{-2s}}}, 0\right)$$
$$+\mathrm{i}\left(-\frac{e^{-s}}{\sqrt{4+e^{-2s}}}, 0, -\frac{e^{-2s}}{2\sqrt{4+e^{-2s}}}, \frac{\sqrt{4+e^{-2s}}}{2}\right) \in \mathbb{C}^4,$$

for all $s \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = \left(-1, 0, -\frac{e^{-w}}{2}, \frac{e^{-w}}{2}\right).$$

Therefore, the solution of the Björling problem for the given strip is

$$X(z) = \left(v, u, \frac{1}{2}e^{-u}\sin(v), -\frac{1}{2}e^{-u}\sin(v)\right),$$

with $z = u + \mathrm{i}v \in \mathbb{C}$. Note that x^2 is an axis of symmetry of the complete spacelike, zero mean curvature surface $S = X(\mathbb{C})$.

Example 4.11. Consider

$$c(s) = (\sinh(s), 0, 0, \cosh(s)) \in \mathbb{L}^4,$$

$$n(s) = (0, \cos(s), \sin(s), 0) + \mathrm{i}(\sinh(s), 0, 0, \cosh(s)) \in \mathbb{C}^4,$$

for all $s \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (0, -\sin(w), \cos(w), 0).$$

Therefore, the solution of the Björling problem for the given strip is the complete spacelike, zero mean curvature surface

$$X(z) = \begin{bmatrix} \cosh(u) & 0 & 0 & \sinh(u) \\ 0 & \cos(u) & -\sin(u) & 0 \\ 0 & \sin(u) & \cos(u) & 0 \\ \sinh(u) & 0 & 0 & \cosh(u) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\sinh(v) \\ \cos(v) \end{bmatrix},$$

with $z = u + \mathrm{i}v \in \mathbb{C}$. Note that $\Pi^2 = [e_1, e_4]$ is a timelike 2-plane of symmetry of the surface $S = X(\mathbb{C})$.

Example 4.12. Consider

$$c(s) = (\cos(s), \sin(s), 0, 0) \in \mathbb{L}^4,$$

$$n(s) = (\cos(s), \sin(s), 0, 0) + \mathrm{i}(0, 0, \sinh(s), \cosh(s)) \in \mathbb{C}^4,$$

for all $s \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{I}m(n(w)), c'(w)) = (0, 0, -\cosh(w), -\sinh(w), 0).$$

Therefore, the solution of the Björling problem for the given strip is the complete spacelike, zero mean curvature surface

$$X(z) = \begin{bmatrix} \cos(u) & -\sin(u) & 0 & 0 \\ \sin(u) & \cos(u) & 0 & 0 \\ 0 & 0 & \cosh(u) & \sinh(u) \\ 0 & 0 & \sinh(u) & \cosh(u) \end{bmatrix} \begin{bmatrix} \cosh(v) \\ 0 \\ \sin(v) \\ 0 \end{bmatrix},$$

with $z = u + \mathrm{i}v \in \mathbb{C}$. Note that $\Pi^2 = [e_1, e_2]$ is a spacelike 2-plane of symmetry of the surface $S = X(\mathbb{C})$.

Example 4.13. Consider

$$c(s) = (s^2, s, 0, s^2) \in \mathbb{L}^4,$$

$$n(s) = \frac{1}{\sqrt{2 + 4s^2}}\{(1, -2s, -1, 0) + \mathrm{i}(1 + 4s^2, 2s, 1, 2 + 4s^2)\} \in \mathbb{C}^4,$$

for all $s \in \mathbb{R}$. Calculating as above, we obtain that

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (-1, 0, -1, -1).$$

The solution of the Björling problem for the given strip is the complete spacelike, zero mean curvature surface

$$X(z) = (u^2 - v^2 + v, u, v, u^2 - v^2 + v),$$

with $z = u + \mathrm{i}v \in \mathbb{C}$. This surface intersects the degenerate 2-plane $\Pi^2 = [e_1 + e_4, e_2]$ orthogonally along $X(u, 0) = c(u)$, but Π^2 is not a plane of symmetry of S. On the other hand, if we take again the curve $c(s) = (s^2, s, 0, s^2)$, but take

$$n(s) = \frac{1}{\sqrt{1 + 4s^2}}\{(1, -2s, 0, 0) + \mathrm{i}(4s^2, 2s, 0, 1 + 4s^2)\},$$

this time, we obtain

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (0, 0, -1, 0)$$

and

$$X(z) = (u^2 - v^2, u, v, u^2 - v^2),$$

which is symmetric with respect to the 2-plane Π^2.

Example 4.14. The timelike 3-space $\Pi^3 = \{x^2 = 0\}$ is a 3-space of symmetry of spacelike, zero mean curvature surface S given in Example 3.6.

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